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General solutions of the pseudo-diffusion equation of squeezed states

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Abstract. We show that the projection operator $|pq; ye^{i\varphi}\rangle\langle ye^{i\varphi}; pq|$, where $|pq; ye^{i\varphi}\rangle$ is a squeezed state, obeys a partial differential equation in which the squeeze parameter y plays the role of time. It follows that related functions, such as the probability distribution functions and the Wigner function are solutions of this equation. This equation will be called a pseudo-diffusion equation, because it resembles a diffusion equation in Minkowski space. We give general solutions of the pseudo-diffusion equation, first by the method of separation of variables and then by the Fourier transform method, and discuss the limitations of the latter method. The Fourier method is used to introduce squeezing into the number states, the thermal light and the Wigner function.

1. Introduction

In quantum optics and, more specifically, in the formalism of coherent states [1, 2], the density operator (either for a pure or mixed state) can be mapped onto two distinct phase-space distribution functions: (a) the P-function or *covariant* form

$$P(p,q) = \langle pq \mid \hat{\rho} \mid pq\rangle = \mathrm{Tr}(\hat{\rho}|pq\rangle\langle pq|) \tag{1}$$

where $|pq\rangle$ are the coherent states. The function $P(p,q)$ is non-negative, so that it can be interpreted as a probability density in phase space. (b) The P^{c}-function or *contravariant* form, defined by the operator equality

$$\hat{\rho} = \int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}|pq\rangle P^{\mathrm{c}}(p,q)\langle pq| \tag{2}$$

can assume negative values. Throughout this paper we shall consider $\hbar = 1$ and, the variables and operators turned dimensionless. The terms *covariant* and *contravariant*, which we use, were coined by Berezin [3]. These function are also known under another nomenclature: the covariant function is the Husimi function [4] or Q-distribution, whereas the contravariant form is also called the P-distribution [5]. The Wigner function $W(p,q)$ [6] is related to the above two distributions as follows:

$$W(p,q) = \int \frac{\mathrm{d}p'\,\mathrm{d}q'}{\pi} \exp\{-[(q-q')^2 + (p-p')^2]\} P^{\mathrm{c}}(p',q') \tag{3}$$

$$P(p,q) = \int \frac{\mathrm{d}p'\,\mathrm{d}q'}{\pi} \exp\{-[(q-q')^2 + (p-p')^2]\} W(p',q'). \tag{4}$$

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The multiplication by a Gaussian function followed by an integration leads to a smoothing of the integrand. Therefore, equations (3) and (4) display a sequential smoothing, $P^{c}(p,q) \to W(p,q) \to P(p,q)$. Due to this smoothing process, the covariant P-function is always non-negative even if the corresponding contravariant P^{c}-funtion and the Wigner W-function assume negative values. The main physical differences between these three functions are: the position and linear momentum variables in the Wigner function are the eigenvalues of the position and momentum operators, so, a phase-space point in that function is well defined, as in a classical phase-space function; in the P-function a phase-space point is the average value of the phase space points, weighted with a Gaussian function, that lie inside a fundamental cell of area $\hbar$; whereas a phase-space point in the P^{c}-function is averaged over the fundamental cell with an anti-Gaussian (positive argument) weight function. This last function is highly singular and it does not exist as a regular function for pure states, it leads to ultradistributions; however, it may exist as a regular function for mixed states, although being, as a distribution, narrower than its other two partners [7, 8]. Consequently, although these three functions have the same information content of a given quantum state, only the Q-distribution, or P-function, can be interpreted as a true probability distribution function (PDF). As such, it can be used to calculate a classical entropy, as defined by Wehrl [9]:

$$S = -\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} P(p,q) \ln P(p,q). \tag{5}$$

Now, the squeezed states [6, 10] are quantum states for which the variance of one of the two quadratures assumes a value below that obtained by using the coherent states (or the vacuum), whereas the variance of the other quadrature takes a higher value, such that the product of both variances never violates the Heisenberg uncertainty principle. Due to this property, it was proposed that the squeezed states could be useful in the detection of gravitational waves [11], whose intensity is lower than the noise of the electromagnetic vacuum. The squeezed states were produced by several experimental groups [12]. So, it becomes significant to study their properties more thoroughly [13] and to discover new mathematical features for this important class of quantum states, this being the aim of the present paper.

The above mappings, equations (1), (2) of $\hat{\rho}$ to $P(p,q)$ can be extended to the squeezed states, defined as

$$|pq;\zeta\rangle = \mathbf{D}(p,q)\mathbf{S}(\zeta)|0\rangle \qquad \text{where} \quad \zeta \equiv y\mathrm{e}^{\mathrm{i}\varphi} \quad (-\infty < y < \infty) \tag{6}$$

and $|0\rangle$ is the vacuum state, which is the ground state of the dimensionless harmonic oscillator, with position and linear momentum written in units of $(m\omega)^{1/2}$ and $(m\omega)^{-1/2}$ (m and ω are mass and angular frequency).

$$\mathbf{D}(p,q) = \exp[-\mathrm{i}(q\mathbf{P} - p\mathbf{Q})] \tag{7}$$

is the displacement operator which generates the coherent states, and

$$\mathbf{S}(\zeta) = \exp\left[\tfrac{1}{2}\left(\zeta \mathbf{a}^{\dagger 2} - \zeta^{*}\mathbf{a}^{2}\right)\right] \quad \left(\mathbf{a} \equiv \frac{\mathbf{Q} + \imath\mathbf{P}}{\sqrt{2}}\right) \tag{8}$$

is the squeezing operator, where the squeeze parameter y vanishes in the coherent-state limit. It is useful to note that the squeeze operator for general squeezing (boost plus rotation) is related to those of pure boost $\mathbf{S}(y)$ and rotation $\mathbf{R}(\varphi/2)$, as follows:

$$\mathbf{S}(y\mathrm{e}^{\mathrm{i}\varphi}) = \mathbf{R}(\varphi/2)\mathbf{S}(y)\mathbf{R}^{\dagger}(\varphi/2) \tag{9}$$

where

$$\mathbf{S}(y) = \exp\left[-\frac{\mathrm{i}y}{2}(\mathbf{QP} + \mathbf{PQ})\right] \qquad \text{and} \qquad \mathbf{R}(\varphi/2) \equiv \exp\left(-\frac{\mathrm{i}\varphi}{2}\mathbf{a}^{\dagger}\mathbf{a}\right). \tag{10}$$

The above relation follows by using the properties

$$\begin{aligned}\mathbf{Q}_r &= \mathbf{R}^\dagger(\varphi/2)\mathbf{Q}\mathbf{R}(\varphi/2) = \cos(\varphi/2)\mathbf{Q} + \sin(\varphi/2)\mathbf{P}\\ \mathbf{P}_r &= \mathbf{R}^\dagger(\varphi/2)\mathbf{P}\mathbf{R}(\varphi/2) = -\sin(\varphi/2)\mathbf{Q} + \cos(\varphi/2)\mathbf{P}.\end{aligned} \tag{11}$$

Replacing $|pq\rangle$ by $|pq;\zeta\rangle$ in (1) and (2) defines two functions $P(p,q;\zeta)$ and $P^c(p,q;\zeta)$. The Wehrl entropy $S(\zeta)$ corresponding to the squeezed states $|pq;\zeta\rangle$ can be generalized as follows:

$$S(\zeta) = -\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} P(p,q;\zeta)\ln P(p,q;\zeta). \tag{12}$$

Although the squeezed state projector operator (13) is a well-defined quantity, the derivation of the PDF for an arbitrary density operator, as defined in (37), is not, in general, a simple task. Therefore, we shall develop a formalism in order to view the squeezing process as being similar to the time-evolution of a diffusive one: an 'initial' arbitrary unsqueezed PDF is evolved by an integral equation, whose kernel is the squeeezing propagator, obtaining so a y-squeezed PDF, which will continue a positive function, see equation (28). This novel procedure is operationally simpler for handling PDFs, and especially marginal PDFs (to be presented elsewhere), since it permits one to extract and to take advantage of the symmetries of the kernel, and consequently of the PDFs. The essential point to note here is that an arbitrary PDF $P(p,q;y)$ obeys a partial differential equation involving the variables p, q and y, whose formal solution permits one to obtain the integral equation.

So, the content of the paper is organized as follows. In section 2 we derive that differential equation, equation (30), which is valid for any state $\hat{\rho}$, pure or mixed. In section 3 we discuss its solutions using the method of separation of variables. In section 4 we give the solution by the Fourier transform method, which yields the solutions in terms of a kernel that depends on y. These solutions are not as general as those that can be obtained by separation of variables. In subsection 4.1, we derive the kernel without rotation ($\varphi = 0$) and in subsection 4.2 we obtain the kernel, also for rotations. In subsection 4.2 we discuss the symmetries of the kernel. In section 5 we apply the Fourier transform method to three examples. Finally, in section 6 we give a summary.

2. The pseudo-diffusion equation

2.1. The unrotated squeezed states

In this section we derive the partial differential equation (27) for the following elementary projector:

$$\Pi(p,q;y) \equiv |pq,y\rangle\langle y,pq| = \mathbf{D}(p,q)\mathbf{S}(y)|0\rangle\langle 0|\mathbf{S}^\dagger(y)\mathbf{D}^\dagger(p,q). \tag{13}$$

Starting from the translation properties of the displacement operator $\mathbf{D}(p,q)$

$$\widetilde{\mathbf{Q}} \equiv \mathbf{Q} - q = \mathbf{D}(p,q)\mathbf{Q}\mathbf{D}^\dagger(p,q) \tag{14}$$

$$\widetilde{\mathbf{P}} \equiv \mathbf{P} - p = \mathbf{D}(p,q)\mathbf{P}\mathbf{D}^\dagger(p,q) \tag{15}$$

one can prove the following relations [7]:

$$\mathbf{Q}\mathbf{\Pi}(p,q;y) = \left(q - \frac{\mathrm{i}}{2}\frac{\partial}{\partial p} + \frac{\mathrm{e}^{2y}}{2}\frac{\partial}{\partial q}\right)\mathbf{\Pi}(p,q;y) \tag{16}$$

$$\mathbf{P}\mathbf{\Pi}(p,q;y) = \left(p + \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{\mathrm{e}^{-2y}}{2}\frac{\partial}{\partial p}\right)\mathbf{\Pi}(p,q;y) \tag{17}$$

$$\mathbf{\Pi}(p,q;y)\mathbf{Q}=\left(q+\frac{\mathrm{i}}{2}\frac{\partial}{\partial p}+\frac{\mathrm{e}^{2y}}{2}\frac{\partial}{\partial q}\right)\mathbf{\Pi}(p,q;y) \tag{18}$$

$$\mathbf{\Pi}(p,q;y)\mathbf{P}=\left(p-\frac{\mathrm{i}}{2}\frac{\partial}{\partial q}+\frac{\mathrm{e}^{-2y}}{2}\frac{\partial}{\partial p}\right)\mathbf{\Pi}(p,q;y) \tag{19}$$

where the last two relations follow from the first two, by taking the adjoint. From these relations we easily obtain the following commutator and anticommutator relations:

$$\left[\widetilde{\mathbf{Q}},\mathbf{\Pi}(p,q;y)\right]=\left[\mathbf{Q},\mathbf{\Pi}(p,q;y)\right]=-\mathrm{i}\frac{\partial}{\partial p}\mathbf{\Pi}(p,q;y) \tag{20}$$

$$\left[\widetilde{\mathbf{P}},\mathbf{\Pi}(p,q;y)\right]=\left[\mathbf{P},\Pi(p,q;y)\right]=\mathrm{i}\frac{\partial}{\partial q}\mathbf{\Pi}(p,q;y) \tag{21}$$

$$\left\{\widetilde{\mathbf{Q}},\mathbf{\Pi}(p,q;y)\right\}=\mathrm{e}^{2y}\frac{\partial}{\partial q}\mathbf{\Pi}(p,q;y) \tag{22}$$

$$\left\{\widetilde{\mathbf{P}},\mathbf{\Pi}(p,q;y)\right\}=\mathrm{e}^{-2y}\frac{\partial}{\partial p}\mathbf{\Pi}(p,q;y)\,. \tag{23}$$

By differentiating the squeezed state $|pq;y\rangle$ with respect to y, from (6), (14) and (15) we get

$$\begin{aligned}\frac{\partial}{\partial y}|pq;y\rangle&=-\frac{\mathrm{i}}{2}\mathbf{D}(p,q)\{\mathbf{P},\mathbf{Q}\}\mathbf{S}(y)|0\rangle\\&=-\frac{\mathrm{i}}{2}\mathbf{D}(p,q)\{\mathbf{P},\mathbf{Q}\}\mathbf{D}^{\dagger}(p,q)\,\mathbf{D}(p,q)\mathbf{S}(y)|0\rangle=-\frac{\mathrm{i}}{2}\{\widetilde{\mathbf{Q}},\widetilde{\mathbf{P}}\}|pq;y\rangle\,.\end{aligned} \tag{24}$$

Using this relation and its adjoint, we get

$$\frac{\partial}{\partial y}\mathbf{\Pi}(p,q;y)=-\frac{\mathrm{i}}{2}[\{\widetilde{\mathbf{Q}},\widetilde{\mathbf{P}}\},\mathbf{\Pi}(p,q;y)]$$

$$=-\frac{\mathrm{i}}{2}\left(\{\widetilde{\mathbf{Q}},[\widetilde{\mathbf{P}},\mathbf{\Pi}(p,q;y)]\}+\{\widetilde{\mathbf{P}},[\widetilde{\mathbf{Q}},\mathbf{\Pi}(p,q;y)]\}\right) \tag{25}$$

$$=\frac{1}{2}\left(\mathrm{e}^{2y}\frac{\partial^2}{\partial q^2}-\mathrm{e}^{-2y}\frac{\partial^2}{\partial p^2}\right)\mathbf{\Pi}(p,q;y) \tag{26}$$

where we first used the operator identity $[\{\mathbf{A},\mathbf{B}\},\mathbf{C}]=\{\mathbf{A},[\mathbf{B},\mathbf{C}]\}+\{\mathbf{B},[\mathbf{A},\mathbf{C}]\}$ and then substituted equations (20)–(23) in (25) to derive (26).

We can cast (26) in a simpler form by a change of variables:

$$\heartsuit(p,q;\lambda)\,\mathbf{\Pi}(p,q;\lambda)\equiv\left[\frac{\partial}{\partial\lambda}-\frac{1}{4}\left(\frac{\partial^2}{\partial p^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right)\right]\mathbf{\Pi}(p,q;\lambda)=0 \tag{27}$$

where $\lambda\equiv\mathrm{e}^{-2y}$. We see that the linear differential operator $\heartsuit$ depends on the three variables $(p,q;\lambda)$. (Note that the symbol $\heartsuit$ is similar to those used to denote other differential operators, such as the D'Alembertian, $\Box$, and the Laplacian, Δ).

This differential equation for the projector $\mathbf{\Pi}$ enables us to derive the same differential equation for distribution functions that depend linearly on $\mathbf{\Pi}$: The P-function for any density operator $\hat{\rho}$ (pure or mixed) is given by

$$P(p,q;y)=\mathrm{Tr}\left[\hat{\rho}\,\mathbf{\Pi}(p,q;y)\right]\,. \tag{28}$$

Since the operator $\heartsuit$ can be pulled under the trace, the general PDF $P(p, q; y)$ must obey the same differential equation (27) as the projector $\mathbf{\Pi}$:

$$\heartsuit(p, q; \lambda)\; P(p, q; \lambda) \equiv \left[\frac{\partial}{\partial\lambda} - \frac{1}{4}\left(\frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right)\right] P(p, q; \lambda) = 0 \qquad (29)$$

which in the y-variable is

$$\frac{\partial P(p, q; y)}{\partial y} = \frac{1}{2}\left(e^{2y}\frac{\partial^2}{\partial q^2} - e^{-2y}\frac{\partial^2}{\partial p^2}\right) P(p, q; y). \qquad (30)$$

The partial differential equation (29) has been derived earlier by different methods [8]. It was called the *pseudo-diffusion equation*, because (a) it resembles the diffusion equation in two dimensions [14], where the parameter λ plays the role of time, and (b) the coefficients of $\frac{\partial^2}{\partial p^2}$ and $\frac{\partial^2}{\partial q^2}$ have opposite signs. Therefore, this equation describes a diffusive process in the p variable and an infusive one in the q variable *for all* λ. In this way a thin distribution along the p-axis gets continuously deformed into a thin distribution along the q-axis, as λ is increased from 0 to ∞. For $\lambda = 1$ or $y = 0$, the distribution becomes symmetric in the (q, p) directions, if the distribution belongs to a number state, $|n\rangle\langle n|$; in this case one recovers the Poisson distribution of the Glauber coherent states representation, as given in (74).

2.2. *The rotated squeezed states*

We now use (27) to derive a differential equation for the rotated projector $\mathbf{\Pi}(p, q; \zeta)$, with $\varphi \neq 0$. The derivation is based on using equations (9) and (11) to rewrite $|pq; \zeta\rangle$ as follows:

$$|pq; \zeta\rangle = \mathbf{D}(p, q)\mathbf{R}(\varphi/2)\mathbf{S}(y)|0\rangle = \mathbf{R}(\varphi/2)\mathbf{D}(p_r, q_r)\mathbf{S}(y)|0\rangle = \mathbf{R}(\varphi/2)|p_r q_r; y\rangle \qquad (31)$$

where

$$q_r = \cos(\varphi/2)q - \sin(\varphi/2)p \qquad \text{and} \qquad p_r = \sin(\varphi/2)q + \cos(\varphi/2)p\,. \qquad (32)$$

Using (31) and its conjugate, we get

$$\mathbf{\Pi}(p, q; \zeta) := |pq; \zeta\rangle\langle\zeta; pq| = \mathbf{R}(\varphi/2)\,\mathbf{\Pi}(p_r, q_r; y)\,\mathbf{R}^\dagger(\varphi/2). \qquad (33)$$

Since the differential operator $\heartsuit$ commutes with the rotation operator $\mathbf{R}(\varphi/2)$, we immediately get the differential equation for $\mathbf{\Pi}(p, q; \zeta)$, by using $\heartsuit$ with the rotated variables (p_r, q_r):

$$\heartsuit(p_r, q_r; \lambda)\,\mathbf{\Pi}(p, q; \zeta) = \mathbf{R}(\varphi/2)\,[\heartsuit(p_r, q_r; \lambda)\,\mathbf{\Pi}(p_r, q_r; y)]\,\mathbf{R}^\dagger(\varphi/2) = 0\,. \qquad (34)$$

The rotated operator $\heartsuit$ is given by

$$\begin{aligned}\heartsuit(p_r, q_r; \lambda) &= \frac{\partial}{\partial\lambda} - \frac{1}{4}\left[\frac{\partial^2}{\partial p_r^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q_r^2}\right] \\ &= \frac{\partial}{\partial\lambda} - \frac{1}{4}\left[\left(\cos^2\frac{\varphi}{2} - \frac{\sin^2\frac{\varphi}{2}}{\lambda^2}\right)\frac{\partial^2}{\partial p^2} + \left(\sin^2\frac{\varphi}{2} - \frac{\cos^2\frac{\varphi}{2}}{\lambda^2}\right)\frac{\partial^2}{\partial q^2}\right. \\ &\quad \left. - \sin\varphi\left(1 - \frac{1}{\lambda^2}\right)\frac{\partial^2}{\partial p\partial q}\right] \end{aligned} \qquad (35)$$

where we have used

$$\begin{aligned}\left(\frac{\partial}{\partial p_r}\right)^2 &= \left(\cos\frac{\varphi}{2}\frac{\partial}{\partial p} + \sin\frac{\varphi}{2}\frac{\partial}{\partial q}\right)^2 = \cos^2\frac{\varphi}{2}\frac{\partial^2}{\partial p^2} + \sin^2\frac{\varphi}{2}\frac{\partial^2}{\partial q^2} + \sin\varphi\frac{\partial^2}{\partial p\partial q} \\ \left(\frac{\partial}{\partial q_r}\right)^2 &= \left(\cos\frac{\varphi}{2}\frac{\partial}{\partial q} - \sin\frac{\varphi}{2}\frac{\partial}{\partial p}\right)^2 = \sin^2\frac{\varphi}{2}\frac{\partial^2}{\partial p^2} + \cos^2\frac{\varphi}{2}\frac{\partial^2}{\partial q^2} - \sin\varphi\frac{\partial^2}{\partial p\partial q}\,.\end{aligned} \qquad (36)$$

Note that for $\varphi = 0$ we recover (29) from (35).

Using equation (34) and the property of the trace, $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$, we get the following useful equality for the rotated PDF

$$P_\rho(p, q; y, \varphi) := \mathrm{Tr}\left(\hat{\rho}|pq; \zeta\rangle\langle\zeta; pq|\right) = \mathrm{Tr}\left[\hat{\rho}_\mathrm{r}\mathbf{\Pi}(p_\mathrm{r}, q_\mathrm{r}; y)\right] = P_{\rho_\mathrm{r}}(p_\mathrm{r}, q_\mathrm{r}; y, 0) \tag{37}$$

where

$$\hat{\rho}_\mathrm{r} := \mathbf{R}^\dagger\left(\frac{\varphi}{2}\right)\ \hat{\rho}\ \mathbf{R}\left(\frac{\varphi}{2}\right). \tag{38}$$

Finally, since $\hat{\rho}_\mathrm{r}$ is independent of $(p, q; \lambda)$, the above equality (37) yields a pseudo-diffusion equation for the rotated distribution $P_\rho(p, q; \zeta)$:

$$\heartsuit(p_\mathrm{r}, q_\mathrm{r}; \lambda)\ P_\rho(p, q; y\mathrm{e}^{\mathrm{i}\varphi}) = \heartsuit(p_\mathrm{r}, q_\mathrm{r}; \lambda)\ P_{\rho_\mathrm{r}}(p_\mathrm{r}, q_\mathrm{r}; y) = 0. \tag{39}$$

It could be useful to have (39) written in terms of the y-variable:

$$\left(\frac{\partial}{\partial y} - \frac{\cosh 2y}{2}\left[(\tanh 2y - \cos\varphi)\frac{\partial^2}{\partial p^2} + (\tanh 2y + \cos\varphi)\frac{\partial^2}{\partial q^2}\right.\right.$$
$$\left.\left. -4\sin\varphi\frac{\partial^2}{\partial p\partial q}\right]\right)P_\rho(p, q; \zeta) = 0. \tag{40}$$

3. Solution by separation of variables

The pseudo-diffusion equation (29) can be solved by the method of separation of variables, by writing the solution as a product of two functions, $P(p, q; \lambda) = \theta(p; \lambda)\psi(q; \lambda)$, where θ depends only on p and λ, and ψ depends only on q and λ. This gives

$$\begin{aligned}
0 &= \frac{1}{P(p, q; \lambda)}\heartsuit P(p, q; \lambda) \\
&= \frac{1}{\theta(p; \lambda)\psi(q; \lambda)}\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\left[\frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right]\right)\theta(p; \lambda)\psi(q; \lambda) \\
&= \frac{1}{\theta(p; \lambda)}\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta(p; \lambda) - \frac{1}{\psi(q; \lambda)}\left(-\frac{\partial}{\partial\lambda} - \frac{1}{4\lambda^2}\frac{\partial^2}{\partial q^2}\right)\psi(q; \lambda).
\end{aligned} \tag{41}$$

Since the first term in (41) depends only on p and λ, while the second term in (41) depends only on q and λ, we conclude that each of them must be equal to a function of λ only, which we denote by $f(\lambda)$. Therefore, any simultaneous solutions of the following two equations will yield possible solutions of the pseudo-diffusion equation:

$$\left(\frac{\partial}{\partial\lambda} - f(\lambda) - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta(p; \lambda) = 0 \tag{42}$$

$$\left(-\frac{\partial}{\partial\lambda} - f(\lambda) - \frac{1}{4\lambda^2}\frac{\partial^2}{\partial q^2}\right)\psi(q; \lambda) = \frac{1}{\lambda^2}\left(\frac{\partial}{\partial\lambda^{-1}} - \lambda^2 f(\lambda) - \frac{1}{4}\frac{\partial^2}{\partial q^2}\right)\psi(q; \lambda) = 0 \tag{43}$$

where we used $\frac{\partial}{\partial\lambda} = -\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda^{-1}}$. In the next subsection we shall study the solutions of (42) and (43) first for $f(\lambda) \equiv 0$, and then show that the solutions of these equations for $f(\lambda) \neq 0$ do not produce any new solutions of the pseudo-diffusion equation.

3.1. Solutions for $f(\lambda) \equiv 0$

By setting $f \equiv 0$ in equation (42), we obtain a one-dimensional diffusion equation in p, where $\lambda/4$ plays the role of time. Similarly, for $f \equiv 0$, equation (43) yields a diffusion equation in q, but with $\lambda^{-1}/4$ playing the role of time. Hence, to get all the factorizable solutions of the pseudo-diffusion equation, we need only know the solutions of the one-dimensional diffusion equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) g(x,t) = 0. \tag{44}$$

These solutions are known in the literature [14]. Typical solutions of (44) are

$$g(x,t) = \begin{cases} h(\eta; x, t) \equiv \mathrm{e}^{\pm \mathrm{i}\eta x - \eta^2 t} \\ G(x - x', t) \equiv \dfrac{1}{\sqrt{\pi t}} \exp\left[-\dfrac{(x-x')^2}{4t}\right] \\ \Phi_n(x,t) = \widetilde{H}_n(1, -2t, x) = \left(x + 2t\dfrac{\partial}{\partial x}\right)^n \cdot 1 \end{cases} \tag{45}$$

where the h are called the *'heat wave solutions'*, G is the diffusion propagator and $\Phi_n(x,t)$ are the heat polynomials [14]. The latter functions are special cases of the generalized Hermite polynomials $\widetilde{H}_n$ introduced in [15]. In particular, the time-independent solutions

$$g(x,t) = ax + b = a\Phi_1 + b\Phi_0 \qquad \text{where} \quad a, b = \text{constant} \tag{46}$$

correspond to the first two heat polynomials, $n = 0, 1$. Clearly, products of the solutions (45), such as

$$P(p,q;\lambda) = \begin{cases} G(p - p'; \tfrac{1}{4}\lambda)\, G(q - q'; \tfrac{1}{4}\lambda^{-1}) = \dfrac{1}{\pi} \exp[-\lambda^{-1}(p-p')^2 - \lambda(q-q')^2] \\ h(\eta; p, \tfrac{1}{4}\lambda)\, h(\xi; q, \tfrac{1}{4}\lambda^{-1}) = \mathrm{e}^{\mathrm{i}\eta p - \frac{1}{4}\eta^2\lambda}\, \mathrm{e}^{-\mathrm{i}\xi q - \frac{1}{4}\xi^2\lambda^{-1}} \\ \Phi_m(p; \tfrac{1}{4}\lambda)\, \Phi_n(q; \tfrac{1}{4}\lambda^{-1}) \\ \Phi_m(p; \tfrac{1}{4}\lambda)\, h(\xi; q, \tfrac{1}{4}\lambda^{-1}) \\ (ap + b)(cq + d) \qquad a, b, c, d = \text{constant} \end{cases} \tag{47}$$

are perfectly valid mathematical solutions of the pseudo-diffusion equation. However, in the present paper we are mainly interested in solutions which can be interpreted as normalizable PDFs. Therefore, we shall now look for solutions whose integral over the phase space can be made equal to 1, i.e. we shall require $P(p,q;\lambda) \in L^1$. To obtain such normalizable PDFs, the above solutions (47) must be multiplied by smooth weight functions $A(\eta, \xi)$ of η and ξ, which decay rapidly enough at infinity. Examples will be encountered later, when we discuss the Fourier transform method.

We shall see in subsection 5.3 that the first product in (47) is equal to the Wigner function of the projector Π.

3.2. Solutions for $f(\lambda) \neq 0$

It is easy to check that the solutions of (42) for $f(\lambda) \neq 0$ are given by

$$\theta(p;\lambda) = \theta_0(p;\lambda)\, \mathrm{e}^{F(\lambda)} \qquad \text{where} \quad F(\lambda) \equiv \int_1^\lambda \mathrm{d}\lambda'\, f(\lambda') \tag{48}$$

and $\theta_0(p;\lambda)$, is the solution of (42) for $f \equiv 0$:

$$\left(\frac{\partial}{\partial\lambda} - f(\lambda) - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta_0(p;\lambda)\,\mathrm{e}^{F(\lambda)} = \mathrm{e}^{F(\lambda)}\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta_0(p;\lambda) = 0\,. \tag{49}$$

Similarly, we can show that the solutions of (43) are given by

$$\psi(q;\lambda) = \psi_0(q;\lambda)\,\mathrm{e}^{-F(\lambda)} \tag{50}$$

with *the same* $F(\lambda)$, as defined in (48). Thus, the products of $\theta(p;\lambda)$ and $\psi(q;\lambda)$ for $f(\lambda) \neq 0$ do not lead to anything new, since

$$\theta(p,\lambda)\psi(q,\lambda) = \theta_0(p;\lambda)\psi_0(q;\lambda). \tag{51}$$

4. Solution by Fourier transformation

The Fourier transform method is familiar to physicists, since it is used to calculate the propagators of the wave equations and the Schrödinger equation. But the limitations of this method are usually not emphasized in the physics literature. Applying this method in our case allows us to demonstrate some of its limitations, which are often overlooked.

By taking the Fourier transform of the partial differential equation (29), we get an ordinary differential equation in λ alone:

$$\left[\frac{\partial}{\partial\lambda} + \frac{1}{4}\left(\eta^2 - \frac{1}{\lambda^2}\xi^2\right)\right]\widetilde{P}(\eta,\xi;\lambda) = 0 \tag{52}$$

where $\widetilde{P}$ is the Fourier transform of P. It is easy to check that

$$\widetilde{P}(\eta,\xi;\lambda) = A(\eta,\xi)\exp\left[-\tfrac{1}{4}(\eta^2\lambda + \xi^2\lambda^{-1})\right] \tag{53}$$

is a solution of (52) for any function $A(\eta,\xi)$. The general solution of (29) follows immediately by taking the inverse Fourier transform

$$\begin{aligned} P(p,q;\lambda) &= \int \frac{\mathrm{d}\eta\mathrm{d}\xi}{2\pi}\mathrm{e}^{\mathrm{i}(\eta p - \xi q)}\widetilde{P}(\eta,\xi;\lambda) \\ &= \int \frac{\mathrm{d}\eta\,\mathrm{d}\xi}{2\pi}\mathrm{e}^{\mathrm{i}(\eta p-\xi q)}\mathrm{e}^{-\frac{1}{4}(\eta^2\lambda+\xi^2\lambda^{-1})}A(\eta,\xi)\,. \end{aligned} \tag{54}$$

We see that the solution (54) is simply a linear combination of the *heat-wave* solutions given in (47), with an arbitrary weight function $A(\eta,\xi)$. Therefore, the Fourier transform method, which leads to normalized solutions (in the whole phase space) with Gaussian measure, excludes solutions such as the heat polynomials or their superposition, that arise in the method of separation of variables, but which can be normalized (not necessarily in the whole phase space) by introducing a convenient measure or by adequate boundary conditions.

4.1. The kernel K

By calculating $\widetilde{P}$ at two values, λ and μ, of the squeezing parameter, we can eliminate $A(\eta,\xi)$, and obtain the following expression for the kernel K, which connects the Fourier transforms $\widetilde{P}$ at two squeezing parameters:

$$\begin{aligned} K(\eta,\xi;\lambda,\mu) &:= \frac{\widetilde{P}(\eta,\xi;\lambda)}{\widetilde{P}(\eta,\xi;\mu)} \\ &= \mathrm{e}^{-\frac{1}{4}\eta^2(\lambda-\mu)}\mathrm{e}^{-\frac{1}{4}\xi^2(\lambda^{-1}-\mu^{-1})} \equiv \mathrm{e}^{-\frac{1}{4}(\eta^2\Delta+\xi^2\delta)} \end{aligned} \tag{55}$$

where

$$\Delta \equiv \lambda - \mu \qquad \text{and} \qquad \delta \equiv \frac{1}{\lambda} - \frac{1}{\mu}. \tag{56}$$

Substituting (55) in (54), we obtain

$$P(p, q; \lambda) = \int \frac{\mathrm{d}\eta \, \mathrm{d}\xi}{2\pi} \mathrm{e}^{\mathrm{i}(\eta p - \xi q)} K(\eta, \xi; \lambda, \mu) \widetilde{P}(\eta, \xi; \mu). \tag{57}$$

Equation (57) tells us that given a distribution $P(p, q; \mu)$ at a certain squeezing parameter μ, we can calculate the corresponding distribution at another squeezing parameter λ, *provided that the Fourier transform $\widetilde{P}(\eta, \xi; \mu)$ exists.* In particular, by using $\mu = 1$, we can calculate the squeezed PDFs from their Glauber coherent state counterpart.

Note that in the exponent of (55) the coefficients of η^2 and ξ^2 have opposite signs. Therefore, the expression (55) defines a saddle-shaped surface in the three dimensional space, (ξ, η, K), which changes continuously with the variation of the parameter λ $(0 < \lambda < \infty)$. At $\lambda = 1$ the saddle shape becomes a flat plane, defined by $K \equiv 1$.

4.2. The kernel K for the rotated squeezed states

Since the differential equation for a general squeezing is the same as that for a pure boost, but with q_{r} and p_{r} replacing q and p, for the kernel of a general squeezing we get:

$$\begin{aligned} K(\eta, \xi; \lambda, \mu, \varphi) &= K(\eta_{\mathrm{r}}, \xi_{\mathrm{r}}; \lambda, \mu, 0) \\ &= k(\eta_{\mathrm{r}}; \Delta) \cdot k(\xi_{\mathrm{r}}; \delta) = \exp\left[-\tfrac{1}{4}(\eta_{\mathrm{r}}^2 \Delta + \xi_{\mathrm{r}}^2 \delta)\right] \end{aligned} \tag{58}$$

where Δ and δ are defined in (56), and

$$\eta_{\mathrm{r}} = \cos\frac{\varphi}{2}\,\eta - \sin\frac{\varphi}{2}\,\xi \qquad \text{and} \qquad \xi_{\mathrm{r}} = \sin\frac{\varphi}{2}\,\eta + \cos\frac{\varphi}{2}\,\xi\,. \tag{59}$$

Note that (η, ξ) in (59) transform exactly as (q, p) in (32), in order to keep the following symplectic product invariant:

$$\eta_{\mathrm{r}} p_{\mathrm{r}} - \xi_{\mathrm{r}} q_{\mathrm{r}} = \eta p - \xi q\,. \tag{60}$$

Substituting (59) in (58) yields

$$K(\eta, \xi; \lambda, \mu, \varphi) = \exp\left[-\tfrac{1}{8}\left\{(\Delta + \delta)(\eta^2 + \xi^2) + (\Delta - \delta)\left[(\eta^2 - \xi^2)\cos\varphi - 2\xi\eta\sin\varphi\right]\right\}\right] \tag{61}$$

which clearly reduces to (55) for $\varphi = 0$. The kernel (61) satisfies the symmetry

$$K(\eta, \xi; \lambda, \mu, \varphi) = K(\xi, \eta; \lambda, \mu, \pi - \varphi) = K(\eta, \xi; \lambda^{-1}, \mu^{-1}, \pi + \varphi)\,. \tag{62}$$

In particular, we get

$$K(\eta, \xi; \lambda, \mu, 0) = K(\xi, \eta; \lambda, \mu, \pi) \tag{63}$$

so that a rotation by $\varphi = \pi$ is equivalent to the exchange of the momentum and position variables, as expected from (59), where $\eta_{\mathrm{r}} \to -\xi$ and $\xi_{\mathrm{r}} \to \eta$.

For $\mu = 1$ this kernel simplifies to

$$\begin{aligned} K(\eta, \xi; \lambda, \mu = 1, \varphi) = \exp\Bigg[\frac{1 - \lambda^2}{8\lambda}\Bigg\{\left(\frac{\lambda - 1}{\lambda + 1} + \cos\varphi\right)\eta^2 + \left(\frac{\lambda - 1}{\lambda + 1} - \cos\varphi\right)\xi^2 \\ -2\sin\varphi\,\xi\eta\Bigg\}\Bigg]. \end{aligned} \tag{64}$$

As before, we can get the distribution after a general squeezing from that before the squeezing, as follows:

$$P_\rho(p,q;\lambda,\varphi) = \int \frac{\mathrm{d}\eta\,\mathrm{d}\xi}{2\pi} \exp[\mathrm{i}(\eta p - \xi q)] K(\eta,\xi;\lambda,\mu,\varphi) \widetilde{P}_\rho(\eta,\xi;\mu,\varphi). \tag{65}$$

Using (37), we can also write this equation as

$$\begin{aligned} P_\rho(p,q;\lambda,\varphi) &= P_{\rho_\mathrm{r}}(p_\mathrm{r},q_\mathrm{r};\lambda,0) \\ &= \int \frac{\mathrm{d}\eta\,\mathrm{d}\xi}{2\pi} \exp[\mathrm{i}(\eta p_\mathrm{r} - \xi q_\mathrm{r})] K(\eta,\xi;\lambda,\mu,0) \widetilde{P}_{\rho_\mathrm{r}}(\eta,\xi;\mu,0). \end{aligned} \tag{66}$$

Since $\mathbf{R}(\varphi/2)|n\rangle = \exp[-\mathrm{i}\,n\varphi/2]\,|n\rangle$, the density operators $\hat\rho_D$ which are diagonal in the $|n\rangle$ representation, i.e. $\hat\rho_D = \sum_n p_n |n\rangle\langle n|$, will be invariant under rotations, so that $(\hat\rho_D)_\mathrm{r} = \hat\rho_D$ are independent of the angle φ. Therefore, we have

$$P_{\rho_D}(p,q;\lambda,\varphi) = P_{(\rho_D)_\mathrm{r}}(p_\mathrm{r},q_\mathrm{r};\lambda,0) = P_{\rho_D}(p_\mathrm{r},q_\mathrm{r};\lambda,0) \tag{67}$$

which means that for such diagonal density operators the effect of general squeezing is obtained by first boosting by y and then rotating the contours of resultant distribution by an angle $\varphi/2$.

4.3. *Factorization and symmetry properties of the kernel K*

We note that the kernel K is factorizable, as follows:

$$K(\eta,\xi;\lambda,\mu) = k(\eta;\lambda-\mu)\,k\left(\xi;\frac{1}{\lambda}-\frac{1}{\mu}\right) \qquad \text{where} \quad k(x;\Delta) := \mathrm{e}^{-\frac{1}{4}x^2\Delta} \tag{68}$$

and that K does not depend on the difference $\lambda-\mu$ alone, since $\frac{1}{\lambda}-\frac{1}{\mu} = \frac{\mu-\lambda}{\mu\lambda}$ does not depend solely on $\lambda-\mu$.

From (68) we see that the kernel satisfies the following symmetry property:

$$K(\eta,\xi;\lambda,\mu) = K(\xi,\eta;\lambda^{-1},\mu^{-1}). \tag{69}$$

This symmetry of K allows us to prove the following statement.

Statement. If a coherent-state distribution is symmetric in p and q

$$P(p,q;1) = P(q,p;1) \tag{70}$$

then the corresponding squeezed distribution will satisfy the following symmetry for all λ:

$$P(p,q;\lambda) = P(q,p;\lambda^{-1}). \tag{71}$$

To prove (71), we first note that (70) implies $\widetilde{P}(\eta,\xi;1) = \widetilde{P}(\xi,\eta;1)$ and therefore

$$\begin{aligned} \widetilde{P}(\eta,\xi;\lambda) = K(\eta,\xi;\lambda,1)\widetilde{P}(\eta,\xi;1) &= K(\xi,\eta;\lambda^{-1},1)\widetilde{P}(\xi,\eta;1) \\ &= \widetilde{P}(\xi,\eta;\lambda^{-1}). \end{aligned} \tag{72}$$

The Fourier transform of (72) finally leads to (71).

As a corollary to the above result, we get the useful result

$$P(p,q;\lambda,\varphi) = P(p_\mathrm{r},q_\mathrm{r};\lambda) = P(q_\mathrm{r},p_\mathrm{r};\lambda^{-1}) = P(q,p;\lambda^{-1},\varphi). \tag{73}$$

5. Applications and examples

As illustrative examples of the application of the Fourier-transform solution (57), we consider the following.

5.1. The number states

The density operator for the pure states is $\hat{\rho}_n = |n\rangle\langle n|$ in the coherent-states representation yields a Poisson distribution [1]:

$$P_n(p,q;1) = |\langle n|pq;1\rangle|^2 = \frac{1}{n!}\left(\frac{p^2+q^2}{2}\right)^n \exp\left[-\frac{p^2+q^2}{2}\right] \qquad n \geqslant 0\,. \tag{74}$$

Its Fourier transform is

$$\widetilde{P}_n(\eta,\xi;1) = L_n\left(\frac{\eta^2+\xi^2}{2}\right)\exp\left[-\frac{\eta^2+\xi^2}{2}\right] \tag{75}$$

where $L_n(z)$ are the Laguerre polynomials. To obtain the squeezed distribution, we first expand $L_n((\eta^2+\xi^2)/2)$ as a sum over the products of the associated Laguerre polynomials $L_m^{-1/2}(z)$ [16]:

$$\widetilde{P}_n(\eta,\xi;1) = \sum_{k=0}^{n} L_{n-k}^{-1/2}\left(\frac{\eta^2}{2}\right) L_k^{-1/2}\left(\frac{\xi^2}{2}\right)\exp\left[-\frac{\eta^2+\xi^2}{2}\right]. \tag{76}$$

Using (55) and (57) to squeeze the p-dependent factors, $\mu = 1$, $\xi = 0$, we get

$$\begin{aligned} f_k(p;\lambda) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\eta}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\eta p}\, L_k^{-1/2}\left(\frac{\eta^2}{2}\right)\exp\left[-\eta^2\frac{\lambda+1}{4}\right] \\ &= \sqrt{\frac{2}{\lambda+1}}\left(\frac{\lambda-1}{\lambda+1}\right)^k L_k^{-1/2}\left(\frac{2p^2}{1-\lambda^2}\right)\exp\left[-\frac{p^2}{\lambda+1}\right] \end{aligned} \tag{77}$$

where we used $(\lambda-1)/4+\frac{1}{2} = (\lambda+1)/4$ to get the first line, and then we modified the integral formula No 7.388.4 in [16] to evaluate the integral. The squeezed PDF is therefore given by

$$\begin{aligned} P_n(p,q;\lambda) &= \sum_{k=0}^{n} f_{n-k}(p;\lambda) f_k(q;\lambda^{-1}) \\ &= \frac{2\lambda^{1/2}}{\lambda+1}\left(\frac{\lambda-1}{\lambda+1}\right)^n \exp\left[-\frac{\lambda q^2+p^2}{\lambda+1}\right] \\ &\quad\times\sum_{k=0}^{n}(-1)^k L_{n-k}^{-1/2}\left(\frac{2p^2}{1-\lambda^2}\right) L_k^{-1/2}\left(\frac{2q^2}{1-\lambda^{-2}}\right). \end{aligned} \tag{78}$$

5.2. Squeezing of thermal light

The distribution of thermal photons is given [17] by

$$\hat{\rho}_{\mathrm{th}} = \frac{1}{\overline{n}+1}\sum_{n=0}^{\infty}\left(\frac{\overline{n}}{\overline{n}+1}\right)^n |n\rangle\langle n| \tag{79}$$

where

$$\overline{n} = \frac{1}{\exp(\hbar\omega/k_B T)-1} \tag{80}$$

is the mean number of photons. The distribution function for this density matrix is

$$P_{\mathrm{th}}(p,q;1) \equiv \mathrm{Tr}[\hat{\rho}_{\mathrm{th}}\Pi(p,q)] = \frac{1}{\overline{n}+1}\sum_{n=0}^{\infty}\left(\frac{\overline{n}}{\overline{n}+1}\right)^n P_n(p,q) \tag{81}$$

$$= \frac{1}{\overline{n}+1}\exp\left[-\frac{p^2+q^2}{2(\overline{n}+1)}\right]. \tag{82}$$

where the last line follows from the expression (74) for $P_n(pq; 1)$. Equation (82) shows a disentanglement between p and q, which can be attributed to the fact that thermalization destroys the p–q correlation.

To calculate the PDF for squeezed thermal light, we first calculate the Fourier transform of $P_{\rm th}(p, q; 1)$ and then multiply it with the kernel. Thus, the Fourier transform of the corresponding squeezed PDF is

$$\widetilde{P}_{\rm th}(\eta, \xi; \lambda) = K(\eta, \xi; \lambda, 1)\widetilde{P}_{\rm th}(\eta, \xi; 1) = \exp\left[-\tfrac{1}{4}\{\eta^2(\lambda + 2\overline{n} + 1) + \xi^2(\lambda^{-1} + 2\overline{n} + 1)\}\right]. \tag{83}$$

Hence, the squeezed PDF is given by

$$\begin{aligned} P_{\rm th}(p, q; \lambda) &= \sqrt{\frac{2}{\lambda + 2\overline{n} + 1}} \exp\left[\frac{-p^2}{\lambda + 2\overline{n} + 1}\right] \cdot \sqrt{\frac{2}{\lambda^{-1} + 2\overline{n} + 1}} \exp\left[\frac{-q^2}{\lambda^{-1} + 2\overline{n} + 1}\right] \\ &= f(p, \lambda) f(q, \lambda^{-1}). \end{aligned} \tag{84}$$

Note that the thermal distribution after a general squeezing, including rotation, follows from the expression (84) by replacing p, q by $p_{\rm r}, q_{\rm r}$:

$$P_{\rm th}(p, q; \lambda, \varphi) = f(p_{\rm r}, \lambda) f(q_{\rm r}, \lambda^{-1}). \tag{85}$$

5.2.1. Increase of entropy due to squeezing. Each of the two factors in (84), $f(p; \lambda)$ and $f(q; \lambda^{-1})$, is a distribution function by itself, i.e.

$$\int_{-\infty}^{\infty} \frac{{\rm d}p}{\sqrt{2\pi}} f(p; \lambda) = \int_{-\infty}^{\infty} \frac{{\rm d}p}{\sqrt{2\pi\alpha}} {\rm e}^{-\frac{p^2}{2\alpha}} = 1 \qquad \text{where} \quad \alpha(\lambda) \equiv \frac{\lambda + 2\overline{n} + 1}{2}. \tag{86}$$

Therefore, the factorization (84) allows us to write the Wehrl entropy [9] of $S_{\rm th}$ as a sum of two 'sub-entropies':

$$\begin{aligned} S_{\rm th}(\lambda) &= -\int \frac{{\rm d}p\,{\rm d}q}{2\pi} P_{\rm th}(p, q; \lambda) \ln P_{\rm th}(p, q; \lambda) \\ &= -\int \frac{{\rm d}p}{\sqrt{2\pi}} f(p; \lambda) \ln f(p; \lambda) - \int \frac{{\rm d}q}{\sqrt{2\pi}} f(q; \lambda^{-1}) \ln f(q; \lambda^{-1}) \\ &= s(\lambda) + s(\lambda^{-1}) = 1 + \tfrac{1}{2} \ln\left[\frac{(\lambda + 2\overline{n} + 1)(\lambda^{-1} + 2\overline{n} + 1)}{4}\right] \end{aligned} \tag{87}$$

where

$$\begin{aligned} s(\lambda) &= -\int \frac{{\rm d}x}{\sqrt{2\pi}} f(x; \lambda) \ln f(x; \lambda) = \int \frac{{\rm d}x}{\sqrt{2\pi}} f(x; \lambda) \left(\frac{x^2}{2\alpha} + \tfrac{1}{2} \ln \alpha\right) \\ &= \tfrac{1}{2} + \tfrac{1}{2} \ln \alpha = \tfrac{1}{2} + \tfrac{1}{2} \ln\left[\frac{\lambda + 2\overline{n} + 1}{2}\right]. \end{aligned} \tag{88}$$

Therefore, $S_{\rm th}(\lambda)$ is an even function of y, so that unless it is equal to a constant, it must have either a maximum or a minimum at $y = 0$ or $\lambda = 1$. Physically it does not make sense that the entropy has a maximum at the unsqueezed situation and therefore, we can conclude, even without any explicit calculations, that $S_{\rm th}(\lambda)$ must have a minimum in $\lambda = 1$. Actual calculations confirm our conclusions: using (88) we get

$$\frac{{\rm d}^2 S_{\rm th}}{{\rm d}\lambda^2}(\lambda = 1) = \frac{2\overline{n} + 1}{(2\overline{n} + 2)^2} > 0. \tag{89}$$

This means that any squeezing must lead to an increase in the entropy of the thermal light. The rate of increase of entropy with λ (for λ close to 1) decreases as the temperature increases: For $T \to \infty$ we have $\bar{n} \to \infty$, so that the rate of increase (89) goes to zero at infinite temperature.

It is interesting to note that

$$S_{\rm th}(\lambda) \geqslant S_{\rm th}(1) = 1 + \ln[1 + \bar{n}] \geqslant 1 \tag{90}$$

in accordance with the inequality for general Wehrl entropy, which was postulated by Wehrl [9] and proved by Lieb [18].

We note that also the entropies of the PDF $P_n(p, q; \lambda)$ have a minimum at $\lambda = 1$, as we showed in [8].

5.3. Squeezing of the Wigner function

The Wigner function of the squeezed projector $\Pi(p, q; \zeta)$ is well known. It is given by [6]:

$$\begin{aligned} W_\Pi(p-k, q-x; \zeta) &\equiv \int_{-\infty}^{\infty} \langle x - a|\Pi(p, q; \zeta)|x + a\rangle \, e^{2iak} \, da \\ &= \frac{1}{\pi} \exp\left[-\lambda^{-1}(p-k)_{\rm r}^2 - \lambda(q-x)_{\rm r}^2\right]. \end{aligned} \tag{91}$$

Nevertheless it is illuminating to derive this expression from the unsqueezed Wigner function by using Fourier transform method, particularly since our derivation will illustrate the use of the *rotated* kernel.

First we note that the above Wigner function satisfies our pseudo-diffusion equation (39). This can be easily seen by explicit differentiation of the expression (91). However, a more interesting proof follows by applying the differential operator $\heartsuit$ on $\mathbf{\Pi}$ inside the integral and using 34):

$$\begin{aligned} &\heartsuit(p_{\rm r}, q_{\rm r}; \lambda) W_\Pi(p-k, q-x; \zeta) \\ &\quad = \int_{-\infty}^{\infty} \langle x - a|(\heartsuit(p_{\rm r}, q_{\rm r}; \lambda)\, \mathbf{\Pi}(p, q; \zeta))|x + a\rangle \, e^{2iak} \, da = 0. \end{aligned} \tag{92}$$

To apply the Fourier transform method, we start with the Wigner function for the unsqueezed projector

$$W_\Pi(p-k, q-x; 1) = \frac{1}{\pi} \exp\left[-(p-k)^2 - (q-x)^2\right] \equiv \frac{1}{\pi} \exp\left[-\tilde{p}^2 - \tilde{q}^2\right] \tag{93}$$

which has the following Fourier transform:

$$\begin{aligned} \widetilde{W}_\Pi(\eta, \xi; 1) &= \int \frac{d\tilde{p} d\tilde{q}}{2\pi} \exp[-i(\eta\tilde{p} - \xi\tilde{q})] W_\Pi(\tilde{p}, \tilde{q}; 1) \\ &= \frac{1}{2\pi} \exp\left[-\frac{\eta^2 + \xi^2}{4}\right] = \frac{1}{2\pi} \exp\left[-\frac{\eta_{\rm r}^2 + \xi_{\rm r}^2}{4}\right] \end{aligned} \tag{94}$$

where we used the invariance of the scalar product $\eta^2 + \xi^2 = \eta_{\rm r}^2 + \xi_{\rm r}^2$ under rotation. The later substitution was done in order to simplify the product of the rotated kernel (58) with $\widetilde{W}$:

$$K(\eta_{\rm r}, \xi_{\rm r}; \lambda, 1, 0)\widetilde{W}_\Pi(\eta, \xi; 1) = \frac{1}{2\pi} \exp\left[-\frac{\lambda\eta_{\rm r}^2 + \lambda^{-1}\xi_{\rm r}^2}{4}\right]. \tag{95}$$

Finally, we obtain the squeezed Wigner function (91) by calculating the inverse Fourier transform of (95), using a change of variables, as follows:

$$
\begin{aligned}
W_{\Pi}(\tilde{p}, \tilde{q}; \zeta) &= \frac{1}{2\pi} \int \frac{\mathrm{d}\eta\, \mathrm{d}\xi}{2\pi} \exp[\mathrm{i}(\eta \tilde{p} - \xi \tilde{q})] \exp\left[-\frac{\lambda \eta_{\mathrm{r}}^2 + \lambda^{-1} \xi_{\mathrm{r}}^2}{4}\right] \\
&= \frac{1}{2\pi} \int \frac{\mathrm{d}\eta_{\mathrm{r}}\, \mathrm{d}\xi_{\mathrm{r}}}{2\pi} \exp[\mathrm{i}(\eta_{\mathrm{r}} \tilde{p}_{\mathrm{r}} - \xi_{\mathrm{r}} \tilde{q}_{\mathrm{r}})] \exp\left[-\frac{\lambda \eta_{\mathrm{r}}^2 + \lambda^{-1} \xi_{\mathrm{r}}^2}{4}\right] \\
&= \frac{1}{\pi} \exp\left[-\lambda^{-1} \tilde{p}_{\mathrm{r}}^2 - \lambda \tilde{q}_{\mathrm{r}}^2\right] = \frac{1}{\pi} \exp\left[-\lambda^{-1} (p - k)_{\mathrm{r}}^2 - \lambda (q - x)_{\mathrm{r}}^2\right] \qquad (96)
\end{aligned}
$$

where we substituted the equality of the symplectic products (60) and used $\mathrm{d}\eta_{\mathrm{r}}\mathrm{d}\xi_{\mathrm{r}} = \mathrm{d}\eta\mathrm{d}\xi$, since the Jacobian of the rotation is 1.

6. Summary

By using commutation and anticommutation relations, we derived partial differential equations for the elementary projector $\Pi(p, q; y\mathrm{e}^{\mathrm{i}\varphi})$, for $\varphi = 0$ and for the rotated states, $\varphi \neq 0$. The PDF $P(p, q; \lambda, \varphi)$ obeys the same differential equations. We called equation (29) a pseudo-diffusion equation, because it resembles a diffusion equation in 'Minkowski space', as we argued in subsection 2.1. This equation is interesting mathematically in its own right, regardless of its application to squeezing.

We solved the pseudo-diffusion equation by two different methods, separation of variables and Fourier transform. The first method is more general, but the second one enables us to calculate a squeezed distribution from a given unsqueezed one. This is done by first calculating the Fourier transform $\widetilde{P}$ of the unsqueezed PDF and then calculating the inverse Fourier transform of the product $K\widetilde{P}$, of the kernel K (55) with $\widetilde{P}$. We illustrated this procedure by three examples: we calculated the squeezed PDF of number states, thermal light and the Wigner function. Along the way we discussed some properties of relevant physical quantities, such as the Wehrl entropy [9, 8].

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